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# UNIFORM HARMONIC APPROXIMATION OF BOUNDED FUNCTIONS

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Abstract. Let $\Omega$ be an open set in $\mathbb{R}^n$ and $E$ be a relatively closed subset of $\Omega$. We characterize those pairs $(\Omega, E)$ which have the following property: every function which is bounded and continuous on $E$ and harmonic on $E^0$ can be uniformly approximated by functions harmonic on $\Omega$. Several related results concerning both harmonic and superharmonic approximation are also established.

## 1. Introduction and results

Let $\Omega$ be an open set in $\mathbb{C}$, let $\Omega^*$ denote its Alexandroff (one point) compactification, and let $\mathrm{Hol}(\Omega)$ denote the collection of holomorphic functions on $\Omega$. Further, let $E$ be a relatively closed subset of $\Omega$ and let $C(E)$ denote the collection of continuous complex-valued functions on $E$.

**Theorem A.** *The following are equivalent*:

(a) *for each* $f$ *in* $C(E) \cap \mathrm{Hol}(E^0)$ *and each positive number* $\varepsilon$ *there exists* $g$ *in* $\mathrm{Hol}(\Omega)$ *such that* $|g - f| < \varepsilon$ *on* $E$;
(b) *for each bounded function* $f$ *in* $C(E) \cap \mathrm{Hol}(E^0)$ *and each positive number* $\varepsilon$ *there exists* $g$ *in* $\mathrm{Hol}(\Omega)$ *such that* $|g - f| < \varepsilon$ *on* $E$;
(c) $\Omega^* \backslash E$ *is connected and locally connected.*

The equivalence of (a) and (c) in Theorem A is the celebrated theorem of Arakelyan (see [1] or [2]). The addition of condition (b) here is due to Stray [16, Theorem 1]. Stray [17, p. 359] subsequently asked whether conditions (a) and (b) of Theorem A remain equivalent when "holomorphic" is replaced by "harmonic" throughout.

The harmonic analogue of Arakelyan's theorem was recently established by the author in [8]: see Theorem B below. One of the results of the present paper (Theorem 2 below) identifies which sets $E$ have the harmonic analogue of property (b) in Theorem A and, in so doing, gives a negative answer to Stray's question in all dimensions. When $n = 2$ it simplifies to a result of Nersesyan [15]. However, the general characterization is more delicate, and new arguments are required to deal with higher dimensions.

From now on $\Omega$ denotes an open set in Euclidean space $\mathbb{R}^n$ ($n \geq 2$) and $E$ is a relatively closed subset of $\Omega$. If $A \subseteq \mathbb{R}^n$, then $C(A)$ denotes the collection of real-valued continuous functions on $A$, and $\mathcal{H}(A)$ (resp. $\mathcal{S}(A)$) is the class of functions which are harmonic (resp. superharmonic) on some open set which contains $A$. We

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say that $A$ is $\Omega$-*bounded* if $\overline{A}$ is a compact subset of $\Omega$, and denote by $\widehat{E}$ the union of $E$ with the $\Omega$-bounded (connected) components of $\Omega\backslash E$. The harmonic analogue of Arakelyan's theorem is as follows. The reader is referred to [8, Theorem 5] for this result, or to [9] for a general exposition of the results in this area. An account of thin sets and the fine topology may be found in Doob [6, 1.XI] or Helms [13, Chapter 10].

**Theorem B.** *Let $\Omega$ be an open set in $\mathbb{R}^n$ and $E$ be a relatively closed subset of $\Omega$. The following are equivalent*:

(a) *for each $u$ in $C(E)\cap\mathcal{H}(E^0)$ and each positive number $\varepsilon$ there exists $\nu$ in $\mathcal{H}(\Omega)$ such that $|\nu-u|<\varepsilon$ on $E$;*

(b) (i) *$\Omega\backslash\widehat{E}$ and $\Omega\backslash E^0$ are thin at the same points of $E$, and*

(ii) *for each compact subset $K$ of $\Omega$ there is a compact subset $L$ of $\Omega$ which contains every $\Omega$-bounded component of $\Omega\backslash(E\cup K)$ whose closure intersects $K$.*

It will become clear from the results below that, if we require the function $u$ in (a) above to be bounded, then condition (b) can be significantly relaxed; that is, approximation is possible on a larger class of sets $E$.

Let $\Omega^*=\Omega\cup\{\mathcal{A}\}$, where $\mathcal{A}$ denotes the Alexandroff point for $\Omega$. If $\omega$ is a connected open subset of $\Omega$, then $\mathcal{A}$ is said to be *accessible from* $\omega$ if there is a continuous function $f:[0,+\infty)\to\omega$ such that $f(t)\to\mathcal{A}$ as $t\to+\infty$. We denote by $\widetilde{E}$ (or, sometimes, by $(E)^{\sim}$) the union of $E$ with the components of $\Omega\backslash E$ from which $\mathcal{A}$ is not accessible. Clearly $\widehat{E}\subseteq\widetilde{E}$, but this inclusion may be strict: see Example 1 below. Our main results on approximation of bounded functions are as follows.

**Theorem 1.** *Let $\Omega$ be an open set in $\mathbb{R}^n$ and $E$ be a relatively closed subset of $\Omega$. The following are equivalent*:

(a) *for each bounded function $u$ in $\mathcal{H}(E)$ and each positive number $\varepsilon$ there exists $\nu$ in $\mathcal{H}(\Omega)$ such that $|\nu-u|<\varepsilon$ on $E$;*

(b) *for each bounded function $u$ in $\mathcal{S}(E)$ and each positive number $\varepsilon$ there exists $\nu$ in $\mathcal{S}(\Omega)$ such that $|\nu-u|<\varepsilon$ on $E$;*

(c) (i) *$\Omega\backslash\widetilde{E}$ and $\Omega\backslash E$ are thin at the same points of $E$, and*

(ii) *for each compact subset $K$ of $\Omega$ there is a compact subset $L$ of $\Omega$ such that $\Omega\backslash\widetilde{E}$ and $\Omega\backslash(\widetilde{E}\cup K)^{\sim}$ are thin at the same points of $E\backslash L$.*

**Theorem 2.** *Let $\Omega$ be an open set in $\mathbb{R}^n$ and $E$ be a relatively closed subset of $\Omega$. The following are equivalent*:

(a) *for each bounded function $u$ in $C(E)\cap\mathcal{H}(E^0)$ and each positive number $\varepsilon$ there exists $\nu$ in $\mathcal{H}(\Omega)$ such that $|\nu-u|<\varepsilon$ on $E$;*

(b) *for each bounded function $u$ in $C(E)\cap\mathcal{S}(E^0)$ and each positive number $\varepsilon$ there exists $\nu$ in $C(\Omega)\cap\mathcal{S}(\Omega)$ such that $|\nu-u|<\varepsilon$ on $E$;*

(c) (i) *$\Omega\backslash\widetilde{E}$ and $\Omega\backslash E^0$ are thin at the same points of $E$, and*

(ii) *for each compact subset $K$ of $\Omega$ there is a compact subset $L$ of $\Omega$ such that $\Omega\backslash\widetilde{E}$ and $\Omega\backslash(\widetilde{E}\cup K)^{\sim}$ are thin at the same points of $E\backslash L$.*

The proof of Theorem 1 (resp. Theorem 2) actually yields a little more: if (c) holds, then any function in $\mathcal{H}(E)$ (resp. in $C(E)\cap\mathcal{H}(E^0)$) which is bounded on $\partial E\cap\Omega$ can be uniformly approximated on $E$ by functions in $\mathcal{H}(\Omega)$.

When $n = 2$ Theorems 1 and 2 simplify to the following slight reformulation of a result of Nersesyan [15].

**Corollary 1.** *Let $\Omega$ be an open set in $\mathbb{R}^2$ and $E$ be a relatively closed subset of $\Omega$. The following are equivalent:*

(a) *for each bounded function $u$ in $C(E) \cap \mathcal{H}(E^0)$ and each positive number $\varepsilon$ there exists $\nu$ in $\mathcal{H}(\Omega)$ such that $|\nu - u| < \varepsilon$ on $E$;*
(b) *for each bounded function $u$ in $\mathcal{H}(E)$ and each positive number $\varepsilon$ there exists $\nu$ in $\mathcal{H}(\Omega)$ such that $|\nu - u| < \varepsilon$ on $E$;*
(c) (i) *$\partial \widetilde{E} = \partial E$, and*
(ii) *for each compact subset $K$ of $\Omega$ there is a compact subset $L$ of $\Omega$ such that $\partial \widetilde{E} \backslash L = \partial((\widetilde{E} \cup K)^{\sim}) \backslash L$.*

The thinness conditions in Theorems 1 and 2 are new, but similar in spirit to those in [8]. That paper contains some illustrative examples which are relevant also in the present context. Below we illustrate condition (c) of Corollary 1 by some further examples.

**Example 1.** Let $\Omega = \mathbb{R}^2$, and for each $m$ in $\mathbb{N}$ let $W_m$ denote the open rectangle $((m+1)^{-1}, m^{-1}) \times (0, m)$. We define

$$E_1 = \partial\left(\left(\bigcup_{m=1}^{\infty} W_m\right) \cup (0,1)^2\right), \quad E_2 = \partial\left(\left(\bigcup_{m=1}^{\infty} W_{2m}\right) \cup (0,1)^2\right),$$

and $E_3 = E_2 \cup \{(1/2, 1/2)\}$. Then $\mathbb{R}^2 \backslash E_k$ $(k = 1, 2, 3)$ consists of two components, and the point at infinity is not accessible from the component which contains the point $(1/4, 1/4)$. Clearly $\partial \widetilde{E}_2 = \partial E_2$, but $\partial \widetilde{E}_1 \neq \partial E_1$ and $\partial \widetilde{E}_3 \neq \partial E_3$.

**Example 2.** Let $\Omega = \mathbb{R}^2$, let $W_m$ be as in Example 1, and let

$$F_1 = \partial\left(\left(\bigcup_{m=1}^{\infty} W_m\right) \cup (-\infty,1)^2\right), \quad F_2 = \partial\left(\left(\bigcup_{m=1}^{\infty} W_{2m}\right) \cup (-\infty,1)^2\right),$$

$$F_3 = F_2 \cup \left\{\left(\frac{4m+1}{4m(2m+1)}, m\right) \in \mathbb{R}^2 : m \in \mathbb{N}\right\}.$$

Then condition (c)(ii) of Corollary 1 is satisfied when $E = F_2$. However, it is not satisfied when $E = F_1$ or $E = F_3$, as can be seen by taking $K$ to be $[0,1] \times \{1\}$.

It is now easy to see that condition (c) of Corollary 1 is satisfied when $E = E_2$ and when $E = F_2$ in $\mathbb{R}^2$. However, this is not true of condition (b) in Theorem B. This provides the promised negative answer to the question of Stray recorded above when $n = 2$, and a simple modification of this example gives a counterexample also in higher dimensions, in the light of Theorems B and 2 and of Lemmas 1 and 2 below.

The proofs of Theorems 1 and 2 rely in part on results and techniques of Gauthier, Goldstein and Ow [11], [12], Labrèche [14] and the author [8]. In particular, they implicitly rely on fusion results for harmonic and superharmonic functions. Theorem 1 is proved in §§2–4. Theorem 2 and Corollary 1 are then derived in §5. In §6 we indicate refinements of Theorems 1 and 2 in which better than uniform approximation of bounded functions is achieved.

## 2. Proof of Theorem 1 (c) $\Rightarrow$ (b)

2.1. In this section we collect together some preliminary observations associated with condition (c) of Theorem 1.

**Lemma 1.** *Suppose that $K$ and $L$ are compact subsets of $\Omega$ such that $K \subseteq L$ and such that $\Omega\backslash\widetilde{E}$ and $\Omega\backslash(\widetilde{E} \cup K)^{\sim}$ are thin at the same points of $E\backslash L$. Let $W$ denote the union of the components of $\Omega\backslash(\widetilde{E} \cup K)$ from which $\mathcal{A}$ is not accessible. If $W \neq \varnothing$, then*:

(i) *each point of $(\partial W \cap \Omega)\backslash L$ is regular for the Dirichlet problem on $W$, and*
(ii) *$(\partial W \cap \Omega)\backslash L \subseteq \partial((\widetilde{E} \cup K)^{\sim})$.*

To prove (i), suppose that $z$ is a point of $(\partial W \cap \Omega)\backslash L$ which is irregular for the Dirichlet problem on $W$. Then $\Omega\backslash W$, and hence also $\Omega\backslash(\widetilde{E} \cup K)^{\sim}$, is thin at $z$. However, $\Omega\backslash\widetilde{E}$ contains $W$, and so is nonthin at $z$. Clearly

$$\partial W \cap \Omega \subseteq (\partial\widetilde{E} \cap \Omega) \cup \partial K \subseteq E \cup \partial K,$$

so $z \in E\backslash L$. This contradicts our hypothesis, and thus (i) must hold.

To prove (ii), let $A = (\partial W \cap \Omega)\backslash(L \cup \partial((\widetilde{E} \cup K)^{\sim}))$. Then

$$A \subseteq (\partial(\widetilde{E} \cup K) \cap \Omega)\backslash\partial((\widetilde{E} \cup K)^{\sim}) \subseteq ((\widetilde{E} \cup K)^{\sim})^{0},$$

so $\Omega\backslash(\widetilde{E} \cup K)^{\sim}$ is certainly thin at each point of $A$. Also,

$$A \subseteq (\partial(\widetilde{E} \cup K) \cap \Omega)\backslash L \subseteq E\backslash L,$$

so $\Omega\backslash\widetilde{E}$ is thin at each point of $A$, by hypothesis. Hence $W$ is also thin at each point of $A$. Thus $A$ is a relatively open subset of $\partial W$ which has zero harmonic measure for each component of $W$ (see [6, 1.XI,13]). Each point of $A$ is therefore irregular for the Dirichlet problem on $W$. Since $A \subseteq (\partial W \cap \Omega)\backslash L$, it follows from (i) that $A = \varnothing$. Hence (ii) holds, and the lemma is proved.

**Lemma 2.** *Suppose that $\Omega\backslash\widetilde{E}$ and $\Omega\backslash E$ are thin at the same points of $E$ and let $W = \widetilde{E}\backslash E$. If $W \neq \varnothing$, then*:

(i) *each point of $\partial W \cap \Omega$ is regular for the Dirichlet problem on $W$, and*
(ii) *$\partial W \cap \Omega \subseteq \partial\widetilde{E}$.*

The proof of Lemma 2 is similar to that of Lemma 1, so we omit the details.

Let $\partial^{\infty} A$ denote the boundary of a set $A$ in $\mathbb{R}^n \cup \{\infty\}$. If $W$ is as in Lemma 2, then each point $y$ of $\partial^{\infty} W \cap \partial^{\infty}\Omega$ is regular for the Dirichlet problem on $W$. To see this, we note that, if $y$ were an irregular boundary point of $W$, then $\mathbb{R}^n\backslash W$ would be thin at $y$, and it follows that almost all rays emanating from $y$ lie initially in $W \cup \{y\}$. This is impossible, in view of the definition of $W$.

**Theorem C.** *Let $\omega$ be a connected open subset of $\Omega$ from which $\mathcal{A}$ is not accessible. If $s$ is a subharmonic function on $\omega$ and*

$$\limsup_{x \to y} s(x) \leq a \qquad (y \in \partial\omega \cap \Omega),$$

*then $s \leq a$ on $\omega$.*

Theorem C is implied by a result of Fuglede [7, §4]; an elementary proof may be found in [5].

We will use $H_f^U$ to denote the Perron-Wiener-Brelot solution (when it is defined) of the Dirichlet problem on an open set $U$ with boundary function $f$ defined on $\partial^\infty U$.

2.2. In the next two lemmas we show that, if condition (c)(i) of Theorem 1 holds, then bounded functions in $\mathcal{S}(E)$ can be uniformly approximated on $E$ by functions in $\mathcal{S}(\widetilde{E})$.

**Lemma 3.** *Suppose that $\Omega\backslash\widetilde{E}$ and $\Omega\backslash E$ are thin at the same points of $E$ and let $W = \widetilde{E}\backslash E$. Further, let $F$ be a relatively closed subset of $\Omega$ such that $F \subseteq W$ and $F$ is nonthin at each of its points. Then, for each positive number $\varepsilon$, there is a nonnegative continuous superharmonic function $w$, on an open set $\omega$ which contains $\widetilde{E}$, such that $w \in \mathcal{H}(\omega\backslash F)$ and $w = 1$ on $F$ and $w \leq \varepsilon$ on $E$.*

In proving Lemma 3 we may assume that $F \neq \varnothing$, for otherwise we can choose $w$ to be the zero function. Let $\Omega_0 = \Omega\backslash F$. In this proof $\widehat{R}_s^A$ will denote the regularized reduced function (balayage) of a nonnegative superharmonic function $s$ on $\Omega_0$ relative to a subset $A$ of $\Omega_0$.

Let $\varepsilon \in (0,1)$ and let $(K_m)$ be a sequence of compact subsets of $\Omega$ such that $K_m \subset K_{m+1}^0$ for each $m$ and $\bigcup_m K_m = \Omega$. Further, let

$$L(m,k) = \{x \in \Omega\backslash K_m^0 : \operatorname{dist}(x, \widetilde{E}) \geq k^{-1}\} \qquad (m, k \in \mathbb{N}),$$

where $\operatorname{dist}(x, A)$ denotes the Euclidean distance from a point $x$ to a set $A$. Then

$$\widehat{R}_1^{L(m,k)}(x) \uparrow \widehat{R}_1^{\Omega\backslash(\widetilde{E}\cup K_m^0)}(x) \qquad (x \in \Omega_0; m \in \mathbb{N}; k \to \infty).$$

We temporarily fix $m$ in $\mathbb{N}$. Let $s_m(x)$ denote the above limit and let $D = (\partial W \cap \Omega)\backslash K_m$. Then $s_m = 1$ at points of $D$ where $\Omega\backslash\widetilde{E}$ is nonthin. Let $S$ denote the complementary set of points of $D$. Since $\Omega\backslash\widetilde{E}$ is thin at points of $S$, so also is $\Omega_0\backslash E$, in view of the hypothesis and the fact that $\partial W \cap \Omega \subseteq E$. Hence the smaller set $W\backslash F$ is thin at points of $S$, and so $S$ carries zero harmonic measure for $W\backslash F$. Let $\chi_A$ denote the characteristic function valued 1 on a set $A$ and 0 elsewhere in $\mathbb{R}^n \cup \{\infty\}$. Then

$$s_m(x) \geq H_{\chi_{D\backslash S}}^{W\backslash F}(x) = H_{\chi_D}^{W\backslash F}(x) \qquad (x \in W\backslash F).$$

Let

$$V_m = \{x \in W\backslash(K_{m+1} \cup F) : s_m(x) < 1 - \varepsilon/2\} \cup (F\backslash K_{m+1}).$$

Then $V_m$ is an open set such that $\overline{V}_m \cap \Omega \subset W$ in view of Lemma 2(i). Clearly $s_m \geq 1 - \varepsilon/2$ on $W\backslash(K_{m+1} \cup V_m)$.

Similarly we define

$$L(0,k) = \{x \in \Omega : \operatorname{dist}(x, \widetilde{E}) \geq k^{-1}\} \qquad (k \in \mathbb{N}),$$

$s_0 = \widehat{R}_1^{\Omega\backslash\widetilde{E}}$, and

$$V_0 = \{x \in W\backslash F : s_0(x) < 1 - \varepsilon/2\} \cup F.$$

Let

$$V = \bigcup_{m=0}^{\infty} V_m.$$

Then $V$ is an open set which contains $F$ and which satisfies $\overline{V} \cap \Omega \subseteq W$. It follows from Dini's theorem that there exists $k_0$ in $\mathbb{N}$ such that

$$\widehat{R}_1^{L(0,k_0)}(x) \geq 1 - \varepsilon \qquad (x \in \partial V \cap K_3). \tag{1}$$

Similarly, for each $m$ in $\mathbb{N}$, we can choose $k_m$ in $\mathbb{N}$ such that

$$\widehat{R}_1^{L(m,k_m)}(x) \geq 1 - \varepsilon \qquad (x \in \partial V \cap (K_{m+3} \backslash K_{m+2}^0); m \in \mathbb{N}). \tag{2}$$

Now let

$$L = \bigcup_{m=0}^{\infty} L(m, k_m),$$

let $\omega = \Omega \backslash L$ and

$$w(x) = \begin{cases} H_{\chi_{\partial F \cap \Omega}}^{\Omega \backslash (F \cup L)}(x) & (x \in \Omega \backslash (F \cup L)), \\ 1 & (x \in F). \end{cases}$$

Thus $\omega$ is an open set which contains $\widetilde{E}$, and $w$ is a nonnegative continuous superharmonic function on $\omega$. (It is continuous at points of $\partial F \cap \Omega$ by the nonthinness of $F$ at such points.) Clearly $w \in \mathcal{H}(\omega \backslash F)$. Since $w \leq 1 - \widehat{R}_1^L$ on $\Omega \backslash (F \cup L)$, it follows from (1) and (2) that $w \leq \varepsilon$ on $\partial V \cap \Omega$. Hence, by the maximum principle, we see that $w \leq \varepsilon$ on $\omega \backslash V$ and so on all of $E$.

This completes the proof of Lemma 3.

**Lemma 4.** *Suppose that $\Omega \backslash \widetilde{E}$ and $\Omega \backslash E$ are thin at the same points of $E$, let $u$ be a bounded function in $\mathcal{S}(E)$ and let $\varepsilon > 0$. Then there exists a bounded function $\nu$ in $\mathcal{S}(\widetilde{E})$ such that $|\nu - u| < \varepsilon$ on $E$. Further, if $u$ is continuous, then it can be arranged that $\nu$ is also continuous.*

To see this, let $\Omega, E, u$ and $\varepsilon$ be as in the first sentence of the lemma. (We may assume that $\varepsilon < 1$.) Then there is an open set $V$ such that $E \subseteq V$ and $u \in \mathcal{S}(V)$, and a positive number $a$ such that $|u| < a$ on $V$. If $\widetilde{E} \subseteq V$, then there is nothing to prove. Otherwise, let $W = \widetilde{E} \backslash E$ and let $F$ be a relatively closed subset of $\Omega$ such that $F \subseteq W$ and $W \subseteq V \cup F^0$, and such that $F$ is nonthin at each of its points.

It follows from Lemma 3 that there is a nonnegative continuous superharmonic function $w$, on an open set $\omega$ which contains $\widetilde{E}$, such that $w \in \mathcal{H}(\omega \backslash F)$ and $w = 1$ on $F$ and $w < \varepsilon/(2a+1)$ on $E$. We now define

$$\nu_1(x) = \begin{cases} u(x) & (x \in V \backslash W), \\ \min\{u(x), a + 1 - (2a+1)w(x)\} & (x \in W \backslash F), \\ -a & (x \in F). \end{cases}$$

Since

$$a + 1 - (2a+1)w(x) > a + 1 - \varepsilon > a > u(x) \qquad (x \in \partial W \cap \Omega)$$

and $w \in \mathcal{H}(\omega \backslash F)$, the function $\nu_1$ is superharmonic on $V \backslash F$. Since

$$a + 1 - (2a+1)w(x) = -a < u(x) \qquad (x \in \partial F \cap \Omega),$$

the function $\nu_1$ is equal to $a + 1 - (2a+1)w(x)$ on an open set which contains $F$. Hence the function $\nu$ defined by $\nu = \nu_1 + (2a+1)w$ is bounded and superharmonic on $\omega \cap (V \cup W)$. Further, $u \leq \nu < u + \varepsilon$ on $E$. Finally, if $u$ is continuous, then the above construction clearly results in $\nu$ also being continuous.

2.3. In this section we establish variants of Lemmas 3 and 4 which relate to condition (c)(ii) of Theorem 1. Suppose that condition (c)(ii) of Theorem 1 holds, let $y \in E$ (we dismiss the trivial case where $E = \varnothing$) and $K_0 = \{y\}$. Further, let $(K_m)_{m\geq 1}$ be a sequence of smoothly bounded compact subsets of $\Omega$ such that $K_m \subset K_{m+1}^0$ and $\widehat{K}_m = K_m$ for each $m$ in $\{0, 1, \ldots, \}$ and such that $\bigcup_m K_m = \Omega$. By deleting some members of $(K_m)_{m\geq 1}$ we can arrange, in view of condition (c)(ii), that $\Omega\backslash\widetilde{E}$ and $\Omega\backslash(\widetilde{E} \cup K_m)^\sim$ are thin at the same points of $E\backslash K_{m+1}^0$, for each $m$.

**Lemma 5.** *Suppose that condition* (c)(ii) *of Theorem* 1 *holds and let* $(K_m)$ *be as described above. Further, let* $l \in \mathbb{N}$, *let*

$$U_l = (\widetilde{E} \cup K_l)^\sim \backslash ((\widetilde{E} \cup K_{l-1})^\sim \cup K_{l+2}), \tag{3}$$

*and let* $F$ *be a relatively closed subset of* $\Omega$ *such that* $F \subseteq U_l$ *and* $F$ *is nonthin at each of its points. Then, for each positive number* $\varepsilon$, *there is a nonnegative continuous superharmonic function* $w$, *on an open set* $\omega$ *which contains* $(\widetilde{E}\cup K_{l-1})^\sim \cup (\overline{U}_l \cap \Omega)$, *such that* $w \in \mathcal{H}(\omega\backslash F)$ *and* $w = 1$ *on* $F$ *and* $w \leq \varepsilon$ *on* $(\widetilde{E} \cup K_{l-1})^\sim \cup (\partial U_l \cap \Omega)$.

In proving Lemma 5 we may assume that $F \neq \varnothing$, for otherwise we can choose $w$ to be the zero function. Let $\Omega_0 = \Omega\backslash F$. We again use $\widehat{R}_s^A$ to denote the regularized reduced function of a nonnegative superharmonic function $s$ on $\Omega_0$ relative to a subset $A$ of $\Omega_0$.

Let $\varepsilon \in (0, 1)$ and

$$L(1, k) = \{x \in \Omega\backslash K_{l+1}^0 : \operatorname{dist}(x, (\widetilde{E} \cup K_{l-1})^\sim \cup \overline{U}_l) \geq k^{-1}\} \qquad (k \in \mathbb{N}).$$

It follows from the construction of the sequence $(K_m)$ that $\Omega\backslash((\widetilde{E} \cup K_{l-1})^\sim \cup \overline{U}_l)$ and $\Omega\backslash\widetilde{E}$ are thin at the same points of $E\backslash K_{l+1}^0$. Hence, by the smoothness of $\partial K_{l+2}$, the sets $\Omega\backslash((\widetilde{E} \cup K_{l-1})^\sim \cup \overline{U}_l)$ and $\Omega\backslash\widetilde{E}$ are thin at the same points of $\partial U_l \cap \Omega$. Let

$$s_1(x) = \widehat{R}_1^{\Omega\backslash[(\widetilde{E}\cup K_{l-1})^\sim \cup \overline{U}_l \cup K_{l+1}^0]}(x) \qquad (x \in \Omega_0)$$

and

$$V_1 = \{x \in U_l\backslash F : s_1(x) < 1 - \varepsilon/2\} \cup F.$$

Similarly we define

$$L(m, k) = \{x \in \Omega\backslash K_{l+m}^0 : \operatorname{dist}(x, (\widetilde{E} \cup K_{l-1})^\sim \cup \overline{U}_l) \geq k^{-1}\} \qquad (m \geq 2; k \in \mathbb{N}),$$

$$s_m(x) = \widehat{R}_1^{\Omega\backslash[(\widetilde{E}\cup K_{l-1})^\sim \cup \overline{U}_l \cup K_{l+m}^0]}(x) \qquad (x \in \Omega_0),$$

and

$$V_m = \{x \in U_l\backslash(K_{l+m+1} \cup F) : s_m(x) < 1 - \varepsilon/2\} \cup (F\backslash K_{l+m+1}).$$

Let

$$V = \bigcup_{m=1}^{\infty} V_m.$$

We now argue as in the proof of Lemma 3 (we appeal to Lemma 1 in place of Lemma 2) to see that $V$ is an open set which contains $F$ and which satisfies $\overline{V} \cap \Omega \subseteq U_l$. Since

$$\widehat{R}_1^{L(1,k)}(x) \uparrow s_1(x) \qquad (x \in \Omega_0; k \to \infty),$$

we can choose $k_1$ in $\mathbb{N}$ such that

$$\widehat{R}_1^{L(1,k_1)}(x) \geq 1 - \varepsilon \qquad (x \in \partial V \cap K_{l+4}). \tag{4}$$

Similarly, if $m \geq 2$, we choose $k_m$ in $\mathbb{N}$ such that

$$\widehat{R}_1^{L(m,k_m)}(x) \geq 1 - \varepsilon \qquad (x \in \partial V \cap (K_{l+m+3} \backslash K^0_{l+m+2})). \tag{5}$$

Now let

$$L = \bigcup_{m=1}^{\infty} L(m, k_m),$$

let $\omega = \Omega \backslash L$ and let

$$w(x) = \begin{cases} H^{\Omega\backslash(F\cup L)}_{\chi_{\partial F\cap\Omega}}(x) & (x \in \Omega\backslash(F \cup L)), \\ 1 & (x \in F). \end{cases}$$

Thus $\omega$ is an open set which contains $(\widetilde{E} \cup K_{l-1})^{\sim} \cup (\overline{U}_l \cap \Omega)$, and $w$ is a nonnegative continuous superharmonic function on $\omega$. Clearly $w \in \mathcal{H}(\omega\backslash F)$. Since $w \leq 1 - \widehat{R}_1^L$ on $\Omega\backslash(F \cup L)$, it follows from (4) and (5) that $w \leq \varepsilon$ on $\partial V \cap \Omega$. Hence, by the maximum principle, we see that $w \leq \varepsilon$ on $\omega\backslash V$ and so on $(\widetilde{E} \cup K_{l-1})^{\sim} \cup (\partial U_l \cap \Omega)$.

This completes the proof of Lemma 5.

Lemma 5 will now be used to prove the following.

**Lemma 6.** *Suppose that condition* (c)(ii) *of Theorem* 1 *holds, let* $(K_m)$ *be as described at the beginning of* §2.3, *let* $l \in \mathbb{N}$ *and let* $U_l$ *be as in* (3). *If* $u$ *is a bounded member of* $\mathcal{S}((\widetilde{E} \cup K_{l-1})^{\sim} \cup K_{l+2})$, *and if* $\varepsilon > 0$, *then there exists a bounded member* $\nu$ *of* $\mathcal{S}((\widetilde{E} \cup K_{l-1})^{\sim} \cup (\overline{U}_l \cap \Omega))$ *such that* $|\nu - u| < \varepsilon$ *on* $(\widetilde{E} \cup K_{l-1})^{\sim} \cup (\partial U_l \cap \Omega)$. *Further, if* $u$ *is continuous, then it can be arranged that* $\nu$ *is also continuous.*

To prove this, let $u$ be a bounded member of $\mathcal{S}((\widetilde{E} \cup K_{l-1})^{\sim} \cup K_{l+2})$ and let $\varepsilon \in (0, 1)$. Then there is an open set $V$ such that $(\widetilde{E} \cup K_{l-1})^{\sim} \cup K_{l+2} \subseteq V$ and $u \in \mathcal{S}(V)$, and a positive number $a$ such that $|u| < a$ on $V$. Next, let $F$ be a relatively closed subset of $\Omega$ such that $F \subset U_l$ and $F$ is nonthin at each of its points, and such that $U_l \subseteq V \cup F^0$. By Lemma 5 there is a nonnegative continuous superharmonic function $w$, on an open set $\omega$ which contains $(\widetilde{E} \cup K_{l-1})^{\sim} \cup (\overline{U}_l \cap \Omega)$, such that $w \in \mathcal{H}(\omega\backslash F)$ and $w = 1$ on $F$ and $w < \varepsilon/(2a + 1)$ on $(\widetilde{E} \cup K_{l-1})^{\sim} \cup (\partial U_l \cap \Omega)$.

Let

$$\nu_1(x) = \begin{cases} u(x) & (x \in V\backslash U_l), \\ \min\{u(x), a + 1 - (2a + 1)w(x)\} & (x \in U_l\backslash F), \\ -a & (x \in F). \end{cases}$$

As in the proof of Lemma 4, the function $\nu_1$ is superharmonic on $V\backslash F$, and the function $\nu$ defined by $\nu = \nu_1 + (2a + 1)w$ is bounded and superharmonic on $\omega \cap (V \cup U_l)$. Further,

$$u(x) \leq \nu(x) < u(x) + \varepsilon \qquad (x \in (\widetilde{E} \cup K_{l-1})^{\sim} \cup (\partial U_l \cap \Omega)).$$

Since $(\widetilde{E}\cup K_{l-1})^{\sim}\cup(\overline{U}_l\cap\Omega)\subset\omega\cap(V\cup U_l)$, this completes the proof of Lemma 6. (It is clear that, if $u$ is continuous, then so also is $\nu$.)

2.4. It will now be shown that (c) implies (b) in Theorem 1. Suppose that (c) holds, let $u$ be a bounded function in $\mathcal{S}(E)$ and let $\varepsilon>0$. In view of Lemma 4 there is a bounded function $u_0$ in $\mathcal{S}(\widetilde{E})$ such that $|u_0-u|<2^{-1}\varepsilon$ on $\widetilde{E}$. Let $\varphi_n:[0,+\infty)\to\mathbb{R}\cup\{+\infty\}$ be the function defined by $\varphi_2(t)=\log(1/t)$ or $\varphi_n(t)=t^{2-n}$ if $n\geq 3$. (We interpret $\varphi_n(0)$ as $+\infty$ in either case.) Also, let $(K_l)$ be the sequence of compact subsets of $\Omega$ described at the beginning of §2.3, and let $U_l$ be as in (3). Then $u_0$ is a bounded member of $\mathcal{S}((\widetilde{E}\cup K_0)^{\sim})$, since $K_0=\{y\}\subseteq E$.

We proceed inductively as follows. Suppose that we have a bounded superharmonic function $u_{l-1}$ on an open set $\omega_{l-1}$ which contains $(\widetilde{E}\cup K_{l-1})^{\sim}$, where $l\in\mathbb{N}$. It follows (see [8, Lemma 2] or [9, Lemma 6.2]) that there exist $\nu_{l-1}$ in $\mathcal{S}(\mathbb{R}^n)$, points $x_1,\dots,x_m$ in $\mathbb{R}^n\backslash\omega_{l-1}$ and a nonnegative constant $c$ such that

$$u_{l-1}(x)=\nu_{l-1}(x)-c\sum_{j=1}^{m}\varphi_n(|x-x_j|)$$

on some open set which contains the compact set $K_{l+3}\cap(\widetilde{E}\cup K_{l-1})^{\sim}$. Thus $u_{l-1}$ can be redefined off the set $(\widetilde{E}\cup K_{l-1})^{\sim}$ so that it is superharmonic (and bounded above) on an open set that contains $(\widetilde{E}\cup K_{l-1})^{\sim}\cup K_{l+2}$, apart from finitely many Newtonian (or logarithmic) singularities in $K_{l+2}\backslash(\widetilde{E}\cup K_{l-1})^{\sim}$. Each of these singularities can be joined to $\mathcal{A}$ by a continuous path lying in $\Omega\backslash(\widetilde{E}\cup K_{l-1})^{\sim}$. Hence, by a simple pole-pushing argument (see, for example, [9, §1.6]), there is a bounded member $w_{l-1}$ of $\mathcal{S}((\widetilde{E}\cup K_{l-1})^{\sim}\cup K_{l+2})$ such that $|w_{l-1}-u_{l-1}|<2^{-l-3}\varepsilon$ on $\widetilde{E}\cup K_{l-1}$. It follows from Lemma 6 that there is a bounded member $s_{l-1}$ of $\mathcal{S}((\widetilde{E}\cup K_{l-1})^{\sim}\cup(\overline{U}_l\cap\Omega))$ such that $|s_{l-1}-u_{l-1}|<2^{-l-2}\varepsilon$ on $(\widetilde{E}\cup K_{l-1})^{\sim}$. Finally, as before, we may redefine $s_{l-1}$ off the set $(\widetilde{E}\cup K_{l-1})^{\sim}\cup(\overline{U}_l\cap\Omega)$ so that it is superharmonic on an open set which contains $(\widetilde{E}\cup K_l)^{\sim}$, apart from finitely many Newtonian (or logarithmic) singularities in $K_{l+2}\backslash(\widetilde{E}\cup K_{l-1})^{\sim}$. Each of these singularities can be joined to $\mathcal{A}$ by a continuous path lying in $\Omega\backslash(\widetilde{E}\cup K_{l-1})^{\sim}$, and so we can obtain a bounded member $u_l$ of $\mathcal{S}((\widetilde{E}\cup K_l)^{\sim})$ such that $u_{l-1}\leq u_l<u_{l-1}+2^{-l-1}\varepsilon$ on $(\widetilde{E}\cup K_{l-1})^{\sim}$.

Since $\bigcup_l K_l=\Omega$, the sequence $(u_l)$, which is eventually defined and increasing on any given compact subset of $\Omega$, converges on $\Omega$ to a function $\nu$ in $\mathcal{S}(\Omega)$ such that

$$|\nu-u|\leq\sum_{l=1}^{\infty}(u_l-u_{l-1})+|u_0-u|<\varepsilon\quad\text{on } E.$$

Thus condition (b) of Theorem 1 holds.

## 3. Proof of Theorem 1 (c) ⇒ (a)

3.1. We begin with harmonic analogues of Lemmas 4 and 6.

**Lemma 7.** *Suppose that $\Omega\backslash\widetilde{E}$ and $\Omega\backslash E$ are thin at the same points of $E$, let $u$ be a bounded function in $\mathcal{H}(E)$ and let $\varepsilon>0$. Then there exists a bounded function $\nu$ in $\mathcal{H}(\widetilde{E})$ such that $|\nu-u|<\varepsilon$ on $E$.*

To see this, let $u$ be a bounded function in $\mathcal{H}(E)$ and let $\varepsilon>0$. By Lemma 4 there exist bounded functions $\nu_1$ and $\nu_2$ in $\mathcal{S}(\widetilde{E})$ such that $u<\nu_1<u+\varepsilon$ and

$-u < \nu_2 < -u + \varepsilon$ on $E$. Thus $-\nu_2 < \nu_1$ on $E$, and hence, by Theorem C, on some open set $\omega$ which contains $\widetilde{E}$. Let $\nu$ be the greatest harmonic minorant of $\nu_1$ on $\omega$. Then $-\nu_2 \leq \nu \leq \nu_1$ on $\omega$. Hence $\nu$ is a bounded member of $\mathcal{H}(\widetilde{E})$ and $|\nu - u| < \varepsilon$ on $E$.

**Lemma 8.** *Suppose that condition* (c)(ii) *of Theorem* 1 *holds, let* $(K_m)$ *be as described at the beginning of* §2.3*, and let* $U_l$ *be as in* (3)*. If* $u$ *is a bounded member of* $\mathcal{H}((\widetilde{E} \cup K_{l-1})^{\sim} \cup K_{l+2})$*, where* $l \in \mathbb{N}$*, and if* $\varepsilon > 0$*, then there exists a bounded member* $\nu$ *of* $\mathcal{H}((\widetilde{E} \cup K_{l-1})^{\sim} \cup (\overline{U}_l \cap \Omega))$ *such that* $|\nu - u| < \varepsilon$ *on* $(\widetilde{E} \cup K_{l-1})^{\sim} \cup (\partial U_l \cap \Omega)$.

This can be deduced from Lemma 6 in the same way that Lemma 7 was deduced from Lemma 4.

3.2. It will now be shown that (c) implies (a) in Theorem 1. Suppose that (c) holds, let $u$ be a bounded function in $\mathcal{H}(E)$ and let $\varepsilon > 0$. In view of Lemma 7 there is a bounded function $u_0$ in $\mathcal{H}(\widetilde{E})$ such that $|u_0 - u| < 2^{-1}\varepsilon$ on $\widetilde{E}$. Let $(K_l)$ be the sequence of compact subsets of $\Omega$ described at the beginning of §2.3 and let $U_l$ be as in (3). Then $u_0$ is a bounded member of $\mathcal{H}((\widetilde{E} \cup K_0)^{\sim})$, since $K_0 = \{y\} \subseteq E$.

We proceed inductively as follows. Suppose that $l \in \mathbb{N}$ and that there is a bounded harmonic function $u_{l-1}$ on an open set $\omega_{l-1}$ that contains $(\widetilde{E} \cup K_{l-1})^{\sim}$. It follows from a result of Gauthier, Goldstein and Ow (see [11] and [12], or [9, Theorem 3.5]) that there is a function $\nu_{l-1}$, which is harmonic on $\Omega$ apart from isolated singularities in $\Omega \backslash (E \cup K_{l-1})^{\sim}$, and which satisfies $|\nu_{l-1} - u_{l-1}| < 2^{-l-4}\varepsilon$ on $(\widetilde{E} \cup K_{l-1})^{\sim}$. Only finitely many of the singularities lie in the compact set $K_{l+2}$, and these singularities can be joined to $\mathcal{A}$ by a continuous path lying in $\Omega \backslash (\widetilde{E} \cup K_{l-1})^{\sim}$. A pole-pushing argument now yields a bounded member $w_{l-1}$ of $\mathcal{H}((\widetilde{E} \cup K_{l-1})^{\sim} \cup K_{l+2})$ such that $|w_{l-1} - u_{l-1}| < 2^{-l-3}\varepsilon$ on $\widetilde{E} \cup K_{l-1}$. It follows from Lemma 8 that there is a bounded member $s_{l-1}$ of $\mathcal{H}((\widetilde{E} \cup K_{l-1})^{\sim} \cup (\overline{U}_l \cap \Omega))$ such that $|s_{l-1} - u_{l-1}| < 2^{-l-2}\varepsilon$ on $(\widetilde{E} \cup K_{l-1})^{\sim}$. A further approximation of $s_{l-1}$ by a harmonic function on $\Omega$ with isolated singularities, followed by pole pushing, yields $u_l$ in $\mathcal{H}((\widetilde{E} \cup K_l)^{\sim})$ such that $|u_l - u_{l-1}| < 2^{-l-1}\varepsilon$ on $(\widetilde{E} \cup K_{l-1})^{\sim}$.

Since $\bigcup_l K_l = \Omega$, the sequence $(u_l)$ converges locally uniformly on $\Omega$ to a function $\nu$ in $\mathcal{H}(\Omega)$ such that

$$|\nu - u| \leq \sum_{l=1}^{\infty} |u_l - u_{l-1}| + |u_0 - u| < \varepsilon \quad \text{on } E.$$

Thus condition (a) of Theorem 1 holds.

## 4. Proof of Theorem 1 (a) ⇒ (c) and (b) ⇒ (c)

4.1. The following lemma simultaneously shows that condition (c)(i) is implied by (a) and by (b) in Theorem 1.

**Lemma 9.** *Suppose that, for each bounded* $u$ *in* $\mathcal{H}(E)$ *and each positive number* $\varepsilon$*, there exists* $\nu$ *in* $\mathcal{S}(\widetilde{E})$ *such that* $|\nu - u| < \varepsilon$*. Then* $\Omega \backslash \widetilde{E}$ *and* $\Omega \backslash E$ *are thin at the same points of* $E$.

To prove this we first note that, if $\Omega$ does not have a Green function (so $n = 2$) and $\widetilde{E} = \Omega$, then $E = \Omega$. To see this, suppose otherwise, let $\varepsilon > 0$ and let $u(x) = \log(|x - y|/|x - z|)$, where $y$ and $z$ are distinct points of $\widetilde{E} \backslash E$. Then $u$ is a bounded member of $\mathcal{H}(E)$ and so, by hypothesis, there exists $\nu_\varepsilon$ in $\mathcal{S}(\Omega)$ such that

$|\nu_\varepsilon - u| < \varepsilon$ on $E$. Hence, by Theorem C, $\nu_\varepsilon$ is a lower bounded superharmonic function on $\Omega$, and so is constant. Since $\varepsilon$ can be arbitrarily small, this forces $u$ to be constant on $E$ and so leads to the contradictory conclusion that $E$ is contained in a straight line or circle. We now discount the trivial case where $E = \Omega$. We define $\Omega_0 = \Omega$ if $\Omega$ has a Green function, and $\Omega_0 = \mathbb{R}^2 \backslash B$ otherwise, where $B$ is a closed ball in $\Omega \backslash \widetilde{E}$. Thus, in either case, $\Omega_0$ has a Green function $G(\cdot, \cdot)$. All reduced functions below are with respect to superharmonic functions on $\Omega_0$.

We next claim that, if $U$ is a component of $\widetilde{E} \backslash E$, then each point of $\partial U \cap \Omega$ is regular for the Dirichlet problem on $U$. To see this, let $\varepsilon > 0$, let $y \in U$ and $u(x) = -G(x, y)$. Then there exists $\nu$ in $\mathcal{S}(\widetilde{E})$ such that $|\nu - u| < \varepsilon$ on $E$ and so, by Theorem C, $\nu - u + \varepsilon$ is a positive member of $\mathcal{S}(\widetilde{E})$. Clearly $\nu - u + \varepsilon \geq G_U(y, \cdot)$, where $G_U(y, \cdot)$ is the Green function for $U$ with pole $y$. Hence

$$\limsup_{x \to z} G_U(y, x) \leq 2\varepsilon \qquad (z \in \partial U \cap \Omega).$$

The claim now follows in view of the arbitrary nature of $\varepsilon$.

Suppose that $\Omega \backslash E$ is nonthin at a point $z$ of $E$, and let $\delta$ be a positive number such that $\overline{B(z, \delta)} \subset \Omega_0 \cap \Omega$, where $B(z, \delta)$ denotes the open ball of centre $z$ and radius $\delta$. Further, let $0 < \rho < \delta$, and let $K$ be a compact subset of $B(z, \rho) \backslash E$ such that

$$u_\rho(z) \geq 1/2, \tag{6}$$

where $u_\rho = \widehat{R}_1^K$.

Let $\varepsilon > 0$. Since $-u_\rho$ is a bounded member of $\mathcal{H}(E)$, we know by hypothesis that there exists $\nu_\varepsilon$ in $\mathcal{S}(\widetilde{E})$ such that $|\nu_\varepsilon + u_\rho| < \varepsilon$ on $E$. By lower semicontinuity $\nu_\varepsilon + u_\rho > -\varepsilon$ on some open set which contains $E$, and hence contains $\partial \widetilde{E} \cap \Omega$. We define $V$ to be the union of $B(z, \delta)$ with $\widetilde{E} \backslash E$. Then $\partial V \cap \Omega \subseteq E \cup \partial B(z, \delta)$. Also, $\partial^\infty V \cap \partial^\infty \Omega$ has zero harmonic measure for $V$. (For example, this can be seen from Theorem C if we define $s$ to be the harmonic measure in $V$ of the set $\partial^\infty V \cap \partial^\infty \Omega$, since all points of $\partial V \cap (\Omega \backslash \overline{B(z, \delta)})$ are regular.) Thus, if we define

$$V_m = \{x \in V : \operatorname{dist}(x, \widetilde{E}) < m^{-1}\} \qquad (m \in \mathbb{N}),$$

then, for large values of $m$,

$$-\widehat{R}_{u_\rho}^{\Omega_0 \backslash V_m} - \nu_\varepsilon = H_{-u_\rho}^{V_m} - \nu_\varepsilon \leq H_{-u_\rho - \nu_\varepsilon}^{V_m} \leq \varepsilon \quad \text{on } E \cap B(z, \delta).$$

Hence

$$u_\rho - \widehat{R}_{u_\rho}^{\Omega_0 \backslash V_m} < 2\varepsilon \quad \text{on } E \cap B(z, \delta).$$

If we let $m \to \infty$ and observe that $\varepsilon$ can be arbitrarily small, then we see that

$$\widehat{R}_{u_\rho}^{\Omega_0 \backslash (\widetilde{E} \cap V)}(z) = u_\rho(z). \tag{7}$$

Let $w_\rho = R_1^{B(z, \rho)}$. Then $w_\rho \geq u_\rho$ on $\Omega_0$, so it follows from (6) and (7) that

$$\widehat{R}_{w_\rho}^{\Omega_0 \backslash (\widetilde{E} \cap B(z, \delta))}(z) \geq \widehat{R}_{u_\rho}^{\Omega_0 \backslash (\widetilde{E} \cap V)}(z) = u_\rho(z) \geq \frac{1}{2}.$$

Since $\rho$ can be arbitrarily small, we conclude (see [6, 1.XI.3(a″)]) that $\Omega_0 \backslash (\widetilde{E} \cap B(z, \delta))$, and hence $\Omega \backslash \widetilde{E}$, is nonthin at $z$.

This establishes Lemma 9.

4.2. The following lemma simultaneously shows that condition (c)(ii) is implied by (a) and by (b) in Theorem 1.

**Lemma 10.** *Suppose that, for each bounded function $u$ in $\mathcal{H}(E)$ and each positive number $\varepsilon$, there exists $\nu$ in $\mathcal{S}(\Omega)$ such that $|\nu - u| < \varepsilon$ on $E$. Then condition* (c)(ii) *of Theorem* 1 *holds.*

To prove this, we suppose that condition (c)(ii) fails to hold. Thus there is a compact subset $K$ of $\Omega$ and a sequence $(z_m)$ of distinct points of $E\backslash K$ such that $(z_m)$ converges to a point of $\partial^\infty\Omega$, and such that

(I) $\Omega\backslash\widetilde{E}$ is nonthin at each point $z_m$,

(II) $\Omega\backslash(\widetilde{E}\cup K)^\sim$ is thin at each point $z_m$.

If $\Omega$ does not have a Green function and $(\widetilde{E}\cup K)^\sim = \Omega$, then $\widetilde{E} = \Omega$ and so $E = \Omega$ (see §4.1). We now dismiss the trivial case where $E = \Omega$. We define $\Omega_0 = \Omega$ if $\Omega$ has a Green function, and $\Omega_0 = \mathbb{R}^2\backslash B$ otherwise, where $B$ is a closed ball in $\Omega\backslash(\widetilde{E}\cup K)^\sim$. All reduced functions below are with respect to superharmonic functions on $\Omega_0$. Let $V = (\widetilde{E}\cup K)^\sim\backslash(\widetilde{E}\cup K)$. We recall that each point of $\partial^\infty V \cap \partial^\infty\Omega$ is regular for the Dirichlet problem on $V$. (See the remark preceding Theorem C.)

For each $m$ in $\mathbb{N}$ we choose a positive number $\delta_m$ in $(0, m^{-1})$ such that the balls $\overline{B(z_m, \delta_m)}$ are pairwise disjoint and are contained in $(\Omega_0\cap\Omega)\backslash K$. Further, the numbers $\delta_m$ should be small enough so that, if $W = V\cup(\bigcup_m B(z_m,\delta_m))$, then each point of $\partial^\infty W\cap\partial^\infty\Omega$ is regular for the Dirichlet problem on $W$. (This is possible in view of Wiener's criterion and the corresponding property of $V$.) Next we choose $\eta_m$ in $(0,\delta_m)$ sufficiently small so that the functions $w_m$ defined by

$$w_m = R_1^{B(z_m,\eta_m)}$$

satisfy $w_m < 2^{-m}$ on $\Omega_0\backslash B(z_m,\delta_m)$. In view of property (II) above we can also arrange, by reducing the value of $\eta_m$, if necessary, that

$$\widehat{R}_{w_m}^{\Omega_0\backslash[B(z_m,\delta_m)\cap(\widetilde{E}\cup K)^\sim]}(z_m) < \frac{1}{4} \tag{8}$$

(see [6, 1.XI.3(a″)]).

In view of property (I) above, we can choose (for each $m$) a compact subset $L_m$ of $B(z_m,\eta_m)\backslash\widetilde{E}$ such that

$$u_m(z_m) > 3/4, \tag{9}$$

where

$$u_m = \widehat{R}_{w_m}^{L_m}.$$

Clearly $u_m \in \mathcal{H}(\Omega_0\backslash L_m)$ for each $m$. Since $u_m \le w_m < 2^{-m}$ on $\Omega_0\backslash B(z_m,\delta_m)$, we see that the series $\sum_m u_m$ converges locally uniformly on $\Omega$ to a function $u$ such that $0 \le u < 2$ on $\Omega$ and $u \in \mathcal{H}(\widetilde{E}\cup K)$. By hypothesis there exists $\nu$ in $\mathcal{S}(\Omega)$ such that $|\nu + u| < 1/6$ on $E$. By Theorem C and lower semicontinuity $\nu + u > -1/6$ on an open set $U$ which contains $\widetilde{E}$. Further, we may arrange (by truncation) that $\nu$ is bounded above on $\Omega$, so there is a positive constant $a$ such that $|\nu + u| < a$ on the compact set $K$. Let $f = \chi_K$. Then, in view of our choice of $(\delta_m)$, we see that $H_f^W(x) \to 0$ as $x \to \mathcal{A}$.

We now choose $m'$ large enough such that $H_f^W < (6a)^{-1}$ on $B(z_{m'}, \delta_{m'})$, and define $V' = V \cup B(z_{m'}, \delta_{m'})$ and

$$V_k = \{x \in V' : \operatorname{dist}(x, (\widetilde{E} \cup K)^{\sim}) < k^{-1}\} \qquad (k \in \mathbb{N}).$$

We know (cf. §4.1) that $\partial^\infty V_k \cap \partial^\infty \Omega$ has zero harmonic measure for each component of $V_k$. Hence, for large values of $k$, we see that

$$\begin{aligned} -\widehat{R}_u^{\Omega_0 \backslash V_k}(x) - \nu(x) \le H_{-u-\nu}^{V_k}(x) &\le \frac{1}{6} + aH_f^{V_k}(x) \\ &\le \frac{1}{6} + aH_f^W(x) < \frac{1}{3} \qquad (x \in \widetilde{E} \cap B(z_{m'}, \delta_{m'})), \end{aligned}$$

and so

$$u(x) - \widehat{R}_u^{\Omega_0 \backslash V_k}(x) < \frac{1}{2} \qquad (x \in E \cap B(z_{m'}, \delta_{m'})).$$

Thus, if we let $k \to \infty$, we obtain

$$u(z_{m'}) - \widehat{R}_u^{\Omega_0 \backslash [V' \cap (\widetilde{E} \cup K)^{\sim}]}(z_{m'}) \le \frac{1}{2}. \tag{10}$$

However, since

$$\widehat{R}_u^A = \sum_m \widehat{R}_{u_m}^A$$

for any set $A$ (see [6, 1.VI.3(f)]), we obtain

$$\begin{aligned} u(z_{m'}) - \widehat{R}_u^{\Omega_0 \backslash [V' \cap (\widetilde{E} \cup K)^{\sim}]}(z_{m'}) &\ge u_{m'}(z_{m'}) - \widehat{R}_{u_{m'}}^{\Omega_0 \backslash [V' \cap (\widetilde{E} \cup K)^{\sim}]}(z_{m'}) \\ &> \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \end{aligned}$$

in view of (8) and (9) and the fact that $u_{m'} \le w_{m'}$. This contradicts (10), so condition (c)(ii) must hold.

This completes the proof of Theorem 1.

## 5. Proof of Theorem 2 and Corollary 1

5.1. Theorem 2 follows immediately from Theorem 1 and the following result due to Labrèche and Bensouda and Gauthier (see [14], [10], [4], or see [9, Theorems 3.17 and 6.11]). In this connection we observe that the proof that (c) implies (b) in Theorem 1 (see §2) yields a continuous approximating function $\nu$ if the original function $u$ is continuous.

**Theorem D.** *Let $\Omega$ be an open set in $\mathbb{R}^n$ and $E$ be a relatively closed subset of $\Omega$. The following are equivalent*:

(a) *for each $u$ in $C(E) \cap \mathcal{H}(E^0)$ and each positive number $\varepsilon$ there exists $\nu$ in $\mathcal{H}(E)$ such that $|\nu - u| < \varepsilon$ on $E$*;
(b) *for each $u$ in $C(E) \cap \mathcal{S}(E^0)$ and each positive number $\varepsilon$ there exists a continuous function $\nu$ in $\mathcal{S}(E)$ such that $|\nu - u| < \varepsilon$ on $E$*;
(c) *$\Omega \backslash E$ and $\Omega \backslash E^0$ are thin at the same points of $E$.*

5.2. We will now prove Corollary 1. Let $n = 2$. Clearly (a) implies (b).

Suppose that (b) holds. It follows from Theorem 1 and Lemma 2 that $\partial E \cap \Omega = \partial \widetilde{E} \cap \Omega$, and from the definition of $\widetilde{E}$ that $\partial^\infty E \cap \partial^\infty \Omega = \partial^\infty \widetilde{E} \cap \partial^\infty \Omega$. Hence $\partial E = \partial \widetilde{E}$. Let $K$ be a compact subset of $\Omega$. Then, by Theorem 1, there is a compact subset $L$ of $\Omega$ such that $\Omega \backslash \widetilde{E}$ and $\Omega \backslash (\widetilde{E} \cup K)^\sim$ are thin at the same points of $E \backslash L$. We may assume that $K \subseteq L$. It follows from Lemma 1 that $\partial \widetilde{E} \cap (\Omega \backslash L) = \partial((\widetilde{E} \cup K)^\sim) \cap (\Omega \backslash L)$. As before we deduce that $\partial \widetilde{E} \backslash L = \partial((\widetilde{E} \cup K)^\sim) \backslash L$. Thus condition (c) holds.

Finally, suppose that condition (c) holds, and let $z$ be a point of $E$ at which $\Omega \backslash \widetilde{E}$ is thin. Then, since $n = 2$, there are arbitrarily small circles centered at $z$ which are contained in $\widetilde{E}$. Hence $z \in (\widetilde{E})^0$. By condition (c)(i) it follows that $z \notin \partial E$, so $z \in E^0$ and $\Omega \backslash E^0$ is certainly thin at $z$. This shows that condition (c)(i) of Theorem 2 holds. Let $K$ be a compact subset of $\Omega$. Then, by condition (c)(ii) there is a compact subset $L$ of $\Omega$ such that $\partial \widetilde{E} \backslash L = \partial((\widetilde{E} \cup K)^\sim) \backslash L$. Let $z$ be a point of $E \backslash L$ at which $\Omega \backslash (\widetilde{E} \cup K)^\sim$ is thin. Then, as in the previous paragraph, we see that $z \in ((\widetilde{E} \cup K)^\sim)^0$. Thus $z \notin \partial \widetilde{E}$ and so $z \in (\widetilde{E})^0$. It follows that $\Omega \backslash \widetilde{E}$ is also thin at $z$. Thus condition (c)(ii) of Theorem 2 holds. We can now apply Theorem 2 to see that condition (a) of Corollary 1 holds.

This completes the derivation of Corollary 1 from Theorems 1 and 2.

## 6. A refinement of Theorem 1

A further equivalent condition is added to Theorem 1 by the following result. It yields better-than-uniform approximation near points of $\partial^\infty E \cap \partial^\infty \Omega$ which are regular for the Dirichlet problem on $\Omega$.

**Theorem 3.** *Let $\Omega$ be a connected open set with Green function $G_\Omega(\cdot, \cdot)$, and let $y \in \Omega$. If condition* (c) *of Theorem* 1 *holds, then for each bounded function $u$ in $\mathcal{H}(E)$ and each positive number $\varepsilon$ there exists $\nu$ in $\mathcal{H}(\Omega)$ such that*

$$|\nu(x) - u(x)| < \varepsilon \min\{1, G_\Omega(y, x)\} \qquad (x \in E).$$

The proof of this result is essentially the same as the argument given in §§2, 3. For example, the right-hand side of (1) and (2) should now be replaced by

$$1 - \varepsilon \min\{1, G_\Omega(y, x)\},$$

so that the function $w$ of Lemma 3 satisfies

$$w(x) \leq \varepsilon \min\{1, G_\Omega(y, x)\} \qquad (x \in E).$$

This allows us to arrange that the function $\nu$ of Lemma 4 satisfies

$$|\nu(x) - u(x)| < \varepsilon \min\{1, G_\Omega(y, x)\} \qquad (x \in E).$$

The appropriate pole-pushing estimates and approximation results can be found in [3] or [9, Chapter 3].

A similar refinement of Theorem 2 also holds.

## References


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